Differential Geometry and its Applications 29 (2011) S41–S51

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Contact Lorentzian manifolds

Giovanni Calvaruso [1]

Dipartimento di Matematica "E. De Giorgi", Università del Salento, Lecce, Italy

ARTICLE INFO

Article history:
Available online 22 April 2011
Communicated by O. Kowalski

MSC:
53D10
53C50
53C15

Keywords:
Contact structures
Pseudo-Riemannian metrics
Lorentzian metrics
Curvature
Ricci tensor
Locally symmetric and homogeneous contact pseudo-metric three-manifolds

ABSTRACT

Contact structures with associated pseudo-Riemannian metrics were studied by D. Perrone and the present author (2010) in [8]. We focus here on contact Lorentzian structures, emphasizing their relationship and analogies with respect to the Riemannian case.

1. Introduction

Contact Riemannian structures are a well known and intensively studied research field in differential geometry. The recent monograph of Blair [3] provides a detailed overview of the results obtained in this framework. Contact structures with associated pseudo-Riemannian metrics were studied first by Takahashi in the pioneering work [18]. The case of contact Lorentzian structures (η, g), where η is a contact one-form and g a Lorentzian metric associated to it, has a particular relevance for physics and was considered in [10] and [1]. However, up to our knowledge, the research devoted to the topic essentially concerned the Sasakian case. A systematic study of general contact pseudo-metric structures was undertaken by the present author and D. Perrone in [8].

In this paper, we focus on the relevant case of contact Lorentzian structures. After presenting the technical apparatus needed for further investigations, we prove some general classification results and exhibit several explicit examples. The paper is organized in the following way. Basic formulae for contact Lorentzian manifolds are given in Section 2. $\mathcal{D}$-homothetic deformations of a contact Lorentzian structure will be described in Section 3. In Section 4, we shall establish a fundamental correspondence between Riemannian and Lorentzian metrics associated to the same contact structure. The classification of contact Lorentzian manifolds of constant sectional curvature, in dimension $\geqslant 5$, is reported in Section 5. In Section 6, we deal with the classification of three-dimensional locally symmetric and homogeneous contact Lorentzian manifolds.

E-mail address: giovanni.calvaruso@unisalento.it.

[1] Supported by funds of MURST and the University of Salento.

0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2011.04.006

2. Basic formulae

An *almost contact structure* (φ, ξ, η) on a $(2n+1)$-dimensional smooth manifold M is formed by a $(1,1)$-tensor φ, a global vector field ξ and a 1-form η, such that

$$\begin{aligned} &\text{(i)} \quad \varphi(\xi) = 0, \qquad \eta \circ \varphi = 0, \\ &\text{(ii)} \quad \eta(\xi) = 1, \qquad \varphi^2 = -Id + \eta \otimes \xi \end{aligned} \tag{2.1}$$

and φ has rank $2n$. Let now g denote a Lorentzian metric on M. g is said to be *compatible* with the almost contact structure (φ, ξ, η) if

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y). \tag{2.2}$$

A smooth manifold M, equipped with an almost contact structure (φ, ξ, η) and a compatible Lorentzian metric g, will be called an *almost contact Lorentzian manifold*.

Remark that, by (2.1) and (2.2), $\eta(X) = -g(\xi, X)$. In particular, $g(\xi, \xi) = -1$ and so, the characteristic vector field ξ is time-like. Moreover, (2.2) implies that $g(\varphi X, Y) = -g(X, \varphi Y)$.

Let now M^{2n+1} be an almost contact Lorentzian manifold, endowed with an almost contact structure (φ, ξ, η) and a compatible Lorentzian metric g. In the remaining part of this section, we shall provide the tensorial apparatus which is needed for further studies of almost contact and contact Lorentzian structures. It is worthwhile to compare formulae below with their Riemannian analogues, for which we may refer for example to [3], in order to understand how the different causal character of the Reeb vector fields influences these equations.

Consider $M^{2n+1} \times \mathbb{R}$ and, denoting by $(X, f\frac{d}{dt})$ an arbitrary vector field on such manifold, the almost complex structure defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right).$$

The almost contact structure (φ, ξ, η) is said to be *normal* if and only if the almost complex structure J is integrable. Necessary and sufficient condition for integrability of J is the vanishing of its *Nijenhuis tensor*

$$\begin{aligned} [J, J]\big((X,0),(Y,0)\big) &= -\big([X,Y],0\big) + \left[\left(\varphi X, \eta(X)\frac{d}{dt}\right), \left(\varphi Y, \eta(Y)\frac{d}{dt}\right)\right] \\ &\quad - J\left[\left(\varphi X, \eta(X)\frac{d}{dt}\right), (Y,0)\right] - J\left[(X,0), \left(\varphi Y, \eta(Y)\frac{d}{dt}\right)\right], \\ [J, J]\left((X,0), \left(0, \frac{d}{dt}\right)\right) &= \left[\left(\varphi X, \eta(X)\frac{d}{dt}\right), (-\xi, 0)\right] \\ &\quad - J\left[\left(\varphi X, \eta(X)\frac{d}{dt}\right), \left(0, \frac{d}{dt}\right)\right] - J\big[(X,0),(-\xi,0)\big], \end{aligned}$$

which, expressed in terms of the Nijenhuis tensor of φ, gives

$$[J, J]\big((X,0),(Y,0)\big) = \big(N^{(1)}(X,Y), N^{(2)}(X,Y)\big), \qquad [J, J]\left((X,0), \left(0, \frac{d}{dt}\right)\right) = \big(N^{(3)}(X), N^{(4)}(X)\big),$$

where

$$N^{(1)} = [\varphi, \varphi] + 2\,d\eta \otimes \xi, \qquad N^{(2)}(X,Y) = (\mathcal{L}_{\varphi X}\eta)Y - (\mathcal{L}_{\varphi Y}\eta)X, \qquad N^{(3)} = \mathcal{L}_\xi \varphi, \qquad N^{(4)} = \mathcal{L}_\xi \eta.$$

Moreover, the vanishing of $N^{(1)}$ implies $N^{(2)} = N^{(3)} = N^{(4)} = 0$ [16,17]. Thus, $N^{(1)} = 0$ is a necessary and sufficient condition for the integrability of J. Next, we prove the following.

Lemma 2.1. *Let (φ, ξ, η) be an almost contact structure and g a compatible pseudo-Riemannian metric (that is, one satisfying (2.2)) on M^{2n+1}. Then,*

$$\begin{aligned} 2g\big((\nabla_X \varphi)Y, Z\big) &= 3\,d\Phi(X, \varphi Y, \varphi Z) - 3\,d\Phi(X, Y, Z) + g\big(N^{(1)}(Y,Z), \varphi X\big) \\ &\quad - N^{(2)}(Y,Z)\eta(X) - 2\,d\eta(\varphi Y, X)\eta(Z) + 2\,d\eta(\varphi Z, X)\eta(Y), \end{aligned}$$

for all tangent vector fields X, Y, Z, where we put $\Phi(X,Y) = g(X, \varphi Y)$.

Proof. Starting from the *Koszul formula*

$$\begin{aligned}2g\big((\nabla_X\varphi)Y,Z\big) &= Xg(Y,Z)+Yg(X,Z)-Zg(X,Y)\\ &\quad + g\big([X,Y],Z\big)-g\big([Y,Z],X\big)+g\big([Z,X],Y\big)\end{aligned}$$

and using the fact that, by (2.2), $g(X,Y)=\Phi(\varphi X,Y)-\eta(X)\eta(Y)$, the conclusion follows by a direct calculation. □

Next, if the compatible Lorentzian metric g satisfies

$$g(X,\varphi Y)=(d\eta)(X,Y), \tag{2.3}$$

then η is a contact form on M, ξ the associated *Reeb vector field*, g an *associated metric*, and (M,η,g) (or (M,φ,ξ,η,g)) is called a *contact Lorentzian manifold*.

Suppose from now on that (M^{2n+1},η,g) is a contact Lorentzian manifold. We denote by ∇ the Levi-Civita connection of M. Taking into account (2.1) and (2.3), we have $(d\eta)(\xi,X)=-g(X,\varphi\xi)=0$ and so, denoting by $\mathcal{L}$ the Lie derivative,

$$N^{(4)}=\mathcal{L}_\xi\eta=d\circ i_\xi\eta+i_\xi\,d\eta=d\eta(\xi)+(d\eta)(\xi,\cdot)=0.$$

On the other hand, since $\eta(X)=-g(\xi,X)$, from $\mathcal{L}_\xi\eta=0$ we get

$$0=-(\mathcal{L}_\xi\eta)X=\xi g(\xi,X)-g\big(\xi,[\xi,X]\big)=g(\nabla_\xi\xi,X),$$

for any vector field X. Therefore, $\nabla_\xi\xi=0$, that is, *the integral curves of* ξ *are geodesic*. Moreover, using (2.3), we get

$$\begin{aligned}(\mathcal{L}_{\varphi X}\eta)Y &= (\varphi X)\big(\eta(Y)\big)-\eta[\varphi X,Y]=(\varphi X)\eta(Y)-Y\eta(\varphi X)-\eta[\varphi X,Y]\\ &= 2(d\eta)(\varphi X,Y)=2g(\varphi X,\varphi Y)\end{aligned}$$

and so, $N^{(2)}(X,Y)=2(d\eta)(\varphi X,Y)-2(d\eta)(\varphi Y,X)=0$. Moreover, (2.3) yields at once $d\Phi=0$. Hence, Lemma 2.1 implies the following.

Corollary 2.2. *In a contact Lorentzian manifold* $(M^{2n+1},\varphi,\xi,\eta,g)$,

$$2g\big((\nabla_X\varphi)Y,Z\big)=g\big(N^{(1)}(Y,Z),\varphi X\big)-2\,d\eta(\varphi Y,X)\eta(Z)+2\,d\eta(\varphi Z,X)\eta(Y). \tag{2.4}$$

Next, we remark that on a contact Lorentzian manifold, $N^{(3)}=0$ if and only if ξ is a *Killing vector field*. In fact, taking into account (2.3), from $\mathcal{L}_\xi\eta=0$ we get

$$0=(\mathcal{L}_\xi\,d\eta)(X,Y)=(\mathcal{L}_\xi g)(X,\varphi Y)+g\big(X,(\mathcal{L}_\xi\varphi)Y\big)$$

and so, $\mathcal{L}_\xi g=0$ if and only if $\mathcal{L}_\xi\varphi=0$. This leads to introduce the tensor

$$h=\frac{1}{2}\mathcal{L}_\xi\varphi=\frac{1}{2}N^{(3)}, \tag{2.5}$$

which plays an important role in describing the geometry of a contact Lorentzian manifold. Moreover, using (2.4), the following properties of the covariant derivative can be proved by direct calculation:

$$\nabla_\xi\varphi=0, \tag{2.6}$$

$$\nabla_X\xi=-\varepsilon\varphi X-\varphi hX. \tag{2.7}$$

Exactly as in the Riemannian case, using (2.6) and (2.7), one can easily prove that h is self-adjoint, $h\varphi=-\varphi h$ and $h\xi=\operatorname{tr}h=0$. Moreover, putting $\tau=\mathcal{L}_\xi g$, one has

$$\tau(X,Y)=2g(X,h\varphi Y).$$

Next, a standard orthonormalization process shows that any (almost) contact Lorentzian manifold (M^{2n+1},η,g) admits a special kind of local pseudo-orthonormal basis, called a *φ-basis*. Such a basis is of the form $\{\xi,e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n\}$. We now prove the following.

Lemma 2.3. *In a contact Lorentzian manifold* $(M^{2n+1},\varphi,\xi,\eta,g)$,

$$\operatorname{div}\xi=0,\qquad \operatorname{div}\eta=0.$$

Proof. Consider a local φ-basis $\{\xi, E_1, \dots, E_{2n}\} = \{\xi, e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n\}$ on M^{2n+1}. Then, since $\nabla_\xi \xi = 0$ and $h\varphi = -\varphi h$, using (2.1) and (2.7) we have

$$\operatorname{div} \xi = \operatorname{tr} \nabla \xi = \sum_{i=1}^n g(\nabla_{e_i} \xi, e_i) + \sum_{i=1}^n g(\nabla_{\varphi e_i} \xi, \varphi e_i) = -\sum_{i=1}^n g(\varphi h e_i, e_i) - \sum_{i=1}^n g(\varphi h \varphi e_i, \varphi e_i) = 0.$$

Moreover, we also obtain $\operatorname{div} \eta = -\operatorname{tr} \nabla \eta = -\varepsilon \operatorname{div} \xi = 0$. □

We now recall the following.

Definition 2.4. A contact Lorentzian manifold (M, η, g) is said to be

(i) *Sasakian* if it is normal, that is, $[\varphi, \varphi] + 2\, d\eta \otimes \xi = 0$.
(ii) *K-contact* if $h = 0$, that is, equivalently, ξ is a Killing vector field.

We have the following characterization (see [1,8]).

Theorem 2.5. *An almost contact Lorentzian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is Sasakian if and only if*

$$(\nabla_X \varphi) Y = g(X, Y)\xi + \eta(Y) X. \tag{2.8}$$

In particular, taking $Y = \xi$ in (2.8), we easily get the following.

Corollary 2.6. *A Sasakian Lorentzian manifold is K-contact.*

The converse in Corollary 2.6 does not hold in general, but it holds true in dimension three (see Theorem 4.4).

We now investigate some curvature properties of a contact Lorentzian manifold (M^{2n+1}, η, g). We denote by R its curvature tensor of M, taken with the sign convention $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ (note that this convention is opposite to the one used in [3]). Using (2.7) and $\nabla_\xi \varphi = 0$, we find

$$R(X, \xi)\xi = \nabla_\xi (\varphi X - \varphi h X) + \varphi[X, \xi] - \varphi h[X, \xi] = -\varphi\big((\nabla_\xi h)(X)\big) + \varphi^2 X + h^2 X,$$

that is,

$$\ell := R(\cdot, \xi)\xi = -\varphi(\nabla_\xi h) + \varphi^2 + h^2. \tag{2.9}$$

Next, applying φ to (2.9), we easily get

$$\varphi \ell X = (\nabla_\xi h) X - \varphi X + h^2 \varphi X,$$

which also implies

$$\varphi \ell \varphi X = \big((\nabla_\xi h)\varphi\big) X - \varphi^2 X - h^2 X.$$

So, we proved

$$\ell - \varphi \ell \varphi = 2\big(\varphi^2 + h^2\big). \tag{2.10}$$

Consider now a (local) φ-basis $\{\xi, E_1, \dots, E_{2n}\} = \{\xi, e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n\}$ of vector fields on M. For any index $i = 1, \dots, 2n$, $\{\xi, E_i\}$ spans a time-like plane on the tangent space at each point where the basis is defined. The sectional curvature of these planes is given by

$$K(\xi, e_i) = -R(\xi, e_i, \xi, e_i) = g(\ell e_i, e_i),$$
$$K(\xi, \varphi e_i) = -R(\xi, \varphi e_i, \xi, \varphi e_i) = g(\ell \varphi e_i, \varphi e_i) = -g(\varphi \ell \varphi e_i, e_i).$$

Thus, by (2.10) we get

$$K(\xi, e_i) + K(\xi, \varphi e_i) = -2\big[1 - g\big(h^2 e_i, e_i\big)\big]$$

and so, for the Ricci curvature $\varrho(\xi, \xi) := \operatorname{tr} R(\xi, \cdot)\xi$ in the direction of ξ, we find

$$\varrho(\xi, \xi) = -\sum_{i=1}^n \big(K(\xi, e_i) + K(\xi, \varphi e_i)\big) = 2n - \operatorname{tr} h^2. \tag{2.11}$$

It is well known that K-contact *Riemannian* manifolds are characterized by the condition $\varrho(\xi,\xi)=2n$, since it implies $\operatorname{tr} h^2=0$ and so, $h=0$ because h is diagonalizable (Theorem A of [2]). The same conclusion holds true for contact Lorentzian manifolds. In fact, although a self-adjoint operator in a Lorentzian space needs not to be diagonalizable [11], $h\xi=0$ and the contact distribution $\operatorname{Ker}\eta=\xi^{\perp}$ is space-like (that is, Riemannian). Henceforth, exactly as in the Riemannian case, the tensor h of a contact Lorentzian manifold is diagonalizable. In particular, $\operatorname{tr} h^2=0$ implies $h=0$ and so, we have the following.

Theorem 2.7. *A contact Lorentzian manifold (M,η,g) is K-contact if and only if it satisfies the curvature condition $\varrho(\xi,\xi)=2n$.*

As proved in [8], the situation is much more complicated in general pseudo-Riemannian settings. In fact, any K-contact pseudo-metric manifold satisfies $\varrho(\xi,\xi)=2n-\operatorname{tr} h^2$. However, there exist examples of contact pseudo-metric manifolds for which $\operatorname{tr} h^2=0$ but ξ is not a Killing vector field.

3. $\mathcal{D}$-homothetic deformations

Let (M^{2n+1},η,g) be a contact Lorentzian manifold. Then, it is easy to check that, for any real constant $t>0$, tensors

$$\tilde\eta=t\eta,\qquad \tilde\xi=\frac{1}{t}\xi,\qquad \tilde\varphi=\varphi,\qquad \tilde g=tg-t(t-1)\eta\otimes\eta \tag{3.1}$$

describe another contact pseudo-metric structure on M^{2n+1}, having the same contact distribution $\operatorname{Ker}\tilde\eta=\operatorname{Ker}\eta$, which we call a *$\mathcal{D}$-homothetic deformation of* (φ,ξ,η,g). Clearly, (3.1) is the Lorentzian counterpart of $\mathcal{D}$-homothetic deformations of a contact Riemannian structure [19]. Notice that, by (3.1), $\tilde g(\tilde\xi,X)=-\tilde\eta(X)$. In particular, $\tilde g(\tilde\xi,\tilde\xi)=g(\xi,\xi)=-1$ and so, the $\mathcal{D}$-homothetic deformation of a contact Lorentzian structure is again a contact Lorentzian structure. Thus, $\mathcal{D}$-homothetic deformations can be used to build new examples of contact Lorentzian structures. As we shall see, a contact Lorentzian structure (φ,η,g) and its $\mathcal{D}$-homothetic deformations share the main contact properties, although they can be very different with regard to curvature properties.

We first calculate the Levi-Civita connection $\tilde\nabla$ of $\tilde g$ in terms of g. By (3.1), the Lorentzian metric

$$g'=\frac{1}{t}\tilde g=g-(t-1)\eta\otimes\eta \tag{3.2}$$

is homothetic to $\tilde g$. Therefore, $\tilde\nabla=\nabla'$ and $R'=R$. Using (3.2) and the Koszul formula, a long but straightforward calculation gives

$$\begin{aligned} g'\big(\nabla'_X Y,Z\big)=g(\nabla_X Y,Z)-(t-1)\big\{&\eta(Z)X\big(\eta(Y)\big)-\eta(Z)g(X,\varphi Y)\\ &-\eta(X)g(\varphi Y,Z)-\eta(Y)g(\varphi X,Z)\big\},\end{aligned}$$

from which, again taking into account (3.2) and (2.7), we have

$$\tilde\nabla_X Y=\nabla'_X Y=\nabla_X Y-\frac{t-1}{t}g(hX,\varphi Y)\xi+(t-1)\big\{\eta(X)\varphi Y+\eta(Y)\varphi X\big\}. \tag{3.3}$$

The following result now easily follows from $\tilde h=\frac{1}{2}\mathcal{L}_{\tilde\xi}\tilde\varphi=\frac{1}{2t}\mathcal{L}_{\xi}\varphi=\frac{1}{t}h$ and (3.3).

Theorem 3.1. *A $\mathcal{D}$-homothetic deformation of a K-contact (respectively, Sasakian) Lorentzian structure is again K-contact (respectively, Sasakian).*

We now turn our attention to the curvature of $\tilde g$. We start with the following.

Proposition 3.2. *Let (M,φ,ξ,η,g) be a contact Lorentzian manifold and $(\tilde\varphi,\tilde\xi,\tilde\eta,\tilde g)$ the $\mathcal{D}$-homothetic deformation described by (3.1). Then, for all $X,Y,Z\in\operatorname{Ker}\eta=\operatorname{Ker}\tilde\eta$:*

$$\tilde\ell X=\frac{1}{t^2}\big\{\ell X+\big(t^2-1\big)\varphi^2X+2(t-1)hX\big\}, \tag{3.4}$$

$$\begin{aligned}\tilde R(X,Y)Z={}&R(X,Y)Z-\frac{t-1}{t}\big\{\eta\big(R(X,Y)Z\big)+g\big(X,(\nabla_Y\varphi)Z\big)-g\big(Y,(\nabla_X\varphi)Z\big)\big\}\xi\\ &+\frac{t-1}{t}\big\{g(X,\varphi Z)(-t\varphi Y+\varphi hY)-g(Y,\varphi Z)(-\varphi X+\varphi hX)\big\}\\ &+(t-1)\eta[X,Y]\varphi Z+\frac{t-1}{t}\big\{\eta(\nabla_X Z)\varphi hY-\eta(\nabla_Y Z)\varphi hX\big\}.\end{aligned} \tag{3.5}$$

Proof. To prove Eq. (3.4), we write (2.9) for the contact metric structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$. We then use (3.3) to calculate $\tilde{\nabla}_{\tilde{\xi}} \tilde{h}$ and (2.9) to express $\nabla_\xi h$ in terms of ℓ. Eq. (3.5) follows from (3.3) by a long but straightforward calculation, using the fact that, because of (3.1), $\tilde{g}(X,Y) = tg(X,Y)$ for all $X, Y \in \operatorname{Ker}\eta$. □

Definition 3.3. Let $(M, \varphi, \xi, \eta, g)$ be a contact Lorentzian manifold and $X \in \operatorname{Ker}\eta$. We put

$$K(\xi, X) = \frac{R(X, \xi, X, \xi)}{g(X,X)} = \frac{g(\ell X, X)}{g(X,X)} \quad \text{and} \quad K(X, \varphi X) = \frac{R(X, \varphi X, X, \varphi X)}{g(X,X)^2}.$$

We call $K(\xi, X)$ the *ξ-sectional curvature* determined by X, and $K(X, \varphi X)$ the *φ-sectional curvature* determined by X. We have the following.

Theorem 3.4. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a contact Lorentzian manifold and $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ the $\mathcal{D}$-homothetic deformation described by (3.1). Then, for all $X \in \operatorname{Ker}\eta = \operatorname{Ker}\tilde{\eta}$,*

$$\tilde{K}(\tilde{\xi}, X) = \frac{1}{t^2} K(\xi, X) - \frac{t^2 - 1}{t^2} + 2\frac{t-1}{t^2}\frac{g(hX, X)}{g(X,X)}, \tag{3.6}$$

$$\tilde{K}(X, \varphi X) = \frac{1}{t} K(X, \varphi X) + 3\frac{t-1}{t} + \frac{t-1}{t^2}\frac{g(hX, X)^2 + g(\varphi hX, X)^2}{g(X,X)^2}. \tag{3.7}$$

Moreover, the Ricci tensors and scalar curvatures satisfy

$$\tilde{\varrho} = \varrho + 2(t-1)g + 2(t-1)(nt + n + 1)\eta \otimes \eta + \frac{t-1}{t} g\big(-(\nabla_\xi h)\varphi + 2h, \cdot\big), \tag{3.8}$$

$$\tilde{r} = \frac{1}{t} r + \frac{t-1}{t}\varrho(\xi, \xi) + 2n\frac{(t-1)^2}{t^2}. \tag{3.9}$$

4. Riemannian and Lorentzian metrics associated to the same contact structure

We shall establish a relationship between Riemannian and Lorentzian metrics associated to the same contact structure. Let (φ, ξ, η) be an almost contact structure (respectively, a contact structure) on a smooth manifold M^{2n+1}, and g a compatible (respectively, an associated) *Riemannian* metric. Then, it is easily seen that

$$\bar{g} = g - 2\eta \otimes \eta \tag{4.1}$$

is a *Lorentzian* metric, which, by (2.1) and (2.2), is still compatible with (respectively, associated to) the same almost contact metric structure (φ, ξ, η). Hence, the change of metric described by Eq. (4.1) transforms a compatible Riemannian metric into a Lorentzian one and conversely. Thus, we have the following.

Theorem 4.1. *Let $(M^{2n+1}, \varphi, \xi, \eta)$ be an almost contact manifold. Then, (4.1) determines a one-to-one correspondence between Riemannian and Lorentzian metrics on M compatible with (φ, ξ, η). In particular, if (φ, ξ, η, g) is a contact Riemannian structure, then $(\varphi, \xi, \eta, \bar{g})$ is a contact Lorentzian structure, and conversely.*

Theorem 4.1 clarifies the structure of the set of pseudo-Riemannian metrics associated to the same almost contact structure. Moreover, it permits to obtain the following existence result.

Corollary 4.2. *Any paracompact almost contact manifold $(M^{2n+1}, \varphi, \xi, \eta)$ admits a compatible Lorentzian metric.*

Proof. It is well known that, since M^{2n+1} is paracompact, it admits a Riemannian metric g'. Then,

$$g(X,Y) = \frac{1}{2}\big\{g'(X,Y) + g'(\varphi X, \varphi Y) + \eta(X)\eta(Y)\big\}$$

is a Riemannian metric compatible with (φ, η, g) [3]. We then define $\bar{g}$ by (4.1). Then, $\bar{g}$ is a Lorentzian metric, compatible with the same contact structure. □

Comparing the Levi-Civita connections of (M, η, g) and $(M, \eta, \bar{g})$, we can easily prove the following.

Proposition 4.3. $(M, \eta, \bar{g})$ *is Sasakian (respectively, K-contact) if and only if so is* (M, η, g).

In particular, in dimension three we have the following result.

Theorem 4.4. *A three-dimensional K-contact Lorentzian manifold is Sasakian.*

Proof. It is well known that a three-dimensional K-contact Riemannian manifold is Sasakian (see for example [3]). Suppose now that $(M^3,\eta,\bar{g})$ is a K-contact Lorentzian three-manifold. The Riemannian metric $g=\bar{g}+2\eta\otimes\eta$ is related to g by (4.1) and so, is associated to the same contact structure (φ,ξ,η). Since $(M,\eta,\bar{g})$ is K-contact, by Proposition 4.3 so is (M,η,g). But then, (M^3,η,g) is Sasakian, which implies, again by Proposition 4.3, that $(M,\eta,\bar{g})$ itself is Sasakian. □

With regard to curvature properties, a long but standard calculation proves the following.

Proposition 4.5. *Let (M,φ,ξ,η,g) be a contact Riemannian manifold and $\bar{g}$ the contact Lorentzian metric described by (4.1).*

(a) *For all $X\in\operatorname{Ker}\tilde{\eta}$*

$$\bar{K}(\xi,X)=-K(\xi,X)+4\frac{g(hX,X)}{g(X,X)},\tag{4.2}$$

$$\bar{K}(X,\varphi X)=K(X,\varphi X)+6-2\frac{g(hX,X)^2+g(\varphi hX,X)^2}{g(X,X)^2}.\tag{4.3}$$

(b) *The Ricci tensor and scalar curvature of $\bar{g}$ satisfy*

$$\bar{\varrho}(\xi,\xi)=\varrho(\xi,\xi)=2n-\operatorname{tr}h^2,\tag{4.4}$$

$$\bar{\varrho}(\xi,Y)=\varrho(\xi,Y),\tag{4.5}$$

$$\bar{\varrho}(X,Y)=\varrho(X,Y)+2g(\ell X,Y)+6g(X,Y)+4g(hX,Y)-2g\big(h^2X,Y\big),\tag{4.6}$$

$$\bar{r}=r-2\varrho(\xi,\xi)+8n,\tag{4.7}$$

for all $X,Y\in\operatorname{Ker}\eta$.

We can now use the deformation described in (4.1) to provide a local description of an arbitrary contact Lorentzian structure in dimension three.

Theorem 4.6. *On a three-dimensional contact manifold, in terms of local Darboux coordinates (x,y,z), the Reeb vector field and the contact form are respectively given by $\xi=2\partial_z$ and $\eta=\frac{1}{2}(dz-y\,dx)$, and any Lorentzian associated metric is of the form*

$$g_L=\frac{1}{4}\begin{pmatrix} a-2y^2 & b & y\\ b & c & 0\\ y & 0 & -1\end{pmatrix},\tag{4.8}$$

where a,b,c are smooth functions, such that $ac-b^2-cy^2=1$. In particular, g_L is Sasakian if and only if a,b,c do not depend on z.

Proof. It is well known that $\xi=2\partial_z$ and $\eta=\frac{1}{2}(dz-y\,dx)$ in local Darboux coordinates (see for example [3]). Moreover, as proved in [4], any *Riemannian* associated metric in such coordinates can be expressed in the form

$$g_R=\frac{1}{4}\begin{pmatrix} a & b & -y\\ b & c & 0\\ -y & 0 & 1\end{pmatrix},\tag{4.9}$$

for some smooth functions a,b,c satisfying $ac-b^2-cy^2=1$, and g_R is Sasakian if and only if a,b,c do not depend on z. Now, (4.8) is nothing but the Lorentzian metric obtained applying (4.1)–(4.9). The conclusion then follows from Theorem 4.1 and Proposition 4.3. □

Example 4.7. The standard flat contact Riemannian structure (φ,ξ,η,g_0) of $\mathbb{R}^3(x,y,z)$ is determined in Darboux coordinates by the following tensors:

$$\xi=2\partial_z,\qquad \eta=\frac{1}{2}(dz-y\,dx),\qquad g_0=\frac{1}{4}\begin{pmatrix} 1+y^2+z^2 & z & -y\\ z & 1 & 0\\ -y & 0 & 1\end{pmatrix}\tag{4.10}$$

(see for example [3]). Consider now the Lorentzian metric $g_{0L}=g_0-2\eta\otimes\eta$, obtained applying the deformation (4.1) to the contact Riemannian structure described by (4.10). Then,

$$g_{0L}=\frac{1}{4}\begin{pmatrix} 1-y^2+z^2 & z & y\\ z & 1 & 0\\ y & 0 & -1\end{pmatrix}.$$

It is easy to check, by direct calculation, that the Ricci eigenvalues of g_{0L} are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 8$, the Reeb vector field ξ is a Ricci eigenvector for g_{0L}, corresponding to the Ricci eigenvalue 0, and the scalar curvature is given by $r_{0L} = 8$, according to formulae (4.4)–(4.7). Clearly, $(\varphi, \xi, \eta, g_{0L})$ *is not flat* any more. In particular, $(\varphi, \xi, \eta, g_{0L})$ gives an example of contact Lorentzian three-manifold, not Sasakian, of constant φ-sectional curvature equal to 4. A flat contact Lorentzian structure on $\mathbb{R}^3$ will be described in Section 5.

If we consider the diffeomorphism

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \qquad (x, y, z) \mapsto (x_1, x_2, x_3) = (z\cos x - y\sin x, -z\sin x - y\cos x, -x),$$

then $(f^{-1})^*\eta = \eta_0$ and $(f^{-1})^* g_0 = \bar{g}_{0R}$, where $\eta_0 = \frac{1}{2}(\cos x_3\, dx_1 + \sin x_3\, dx_2)$ and $\bar{g}_{0R} = \frac{1}{4}\sum_i dx_i^2$ [3]. It is evident that the contact metric structure $(\eta_0, \bar{g}_{0R})$ is invariant under the group of translations generated by $(x_i \mapsto x_i + 2\pi)$ and so, it induces a flat contact metric structure on the torus T^3. Applied to this contact metric structure, the deformation (4.1) then gives a contact Lorentzian structure on the compact manifold T^3, which is not flat but has constant φ-sectional curvature equal to 4.

5. Contact Lorentzian manifolds of constant sectional curvature

We first describe a $(2n+1)$-dimensional model of contact Lorentzian manifold of constant sectional curvature. We can refer to [18] for a more general presentation of Sasakian pseudo-metric manifolds of constant sectional curvature.

Example 5.1 *(Sasakian Lorentzian manifolds of constant sectional curvature).* In the pseudo-Euclidean space $(\mathbb{R}_2^{2n+2} \equiv \mathbb{C}_1^{n+1}, \tilde{g}, J)$ with the indefinite standard Kähler structure, we consider the *pseudo-hyperbolic space*

$$\mathbb{H}_1^{2n+1}(-1) = \left\{x \in \mathbb{R}_2^{2n+2}\colon\ \tilde{g}(x, x) = -1\right\},$$

which is a hyperquadric of $\mathbb{R}_2^{2n+2}$, of dimension $(2n+1)$, index 1 and constant sectional curvature -1 [18]. It admits a canonical structure $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ of Sasakian Lorentzian manifold, defined by the following tensors:

$$\bar{g} = \tilde{g}|_{\mathbb{H}_{2s-1}^{2n+1}(-1)}, \qquad \bar{\xi} : x \in \mathbb{H}_{2s-1}^{2n+1}(-1) \mapsto -Jx, \qquad \bar{\eta}(X) = -g(\bar{\xi}, X), \qquad \bar{\varphi} = \bar{\pi} \circ J,$$

where $\bar{\pi}(X) = X + \tilde{g}(X, x)x$. It is well known that $\mathbb{H}_1^{2n+1}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{2n}$ [11] and so, is not simply connected. Its universal covering $\tilde{\mathbb{H}}_1^{2n+1}$ then provides an example of simply connected Sasakian Lorentzian manifold of constant sectional curvature -1.

A first rigidity result concerning Sasakian pseudo-Riemannian manifolds was proved in [1]. In the Lorentzian case, it yields that *any Sasakian Lorentzian connected real hypersurface of* $\mathbb{R}_2^{2n+2}$ *is an open set of* $\mathbb{H}_1^{2n+1}(-1)$.

In Riemannian settings, Olszak [13] obtained a fundamental rigidity result, proving that in any dimension $\geqslant 5$, if a contact Riemannian manifold (M^{2n+1}, η, g) has constant sectional curvature k, then $k = 1$ and (M^{2n+1}, η, g) is Sasakian. This result was extended to pseudo-Riemannian settings in [8]. In particular, in the Lorentzian case, we have the following.

Theorem 5.2. *Let (M^{2n+1}, η, g) be a contact Lorentzian manifold, $n \geqslant 2$. If (M^{2n+1}, g) is of constant sectional curvature k, then $k = -1 = g(\xi, \xi)$ and (M^{2n+1}, η, g) is Sasakian.*

Note that Theorem 5.2 and a result of [18] imply at once the following.

Corollary 5.3. *For any $n \geqslant 2$, the universal covering of the pseudo-hyperbolic space $\mathbb{H}_{2s-1}^{2n+1}(-1)$ is the only simply connected Sasakian Lorentzian manifold of constant sectional curvature.*

Results above leave apart the three-dimensional case. The classification of three-dimensional contact Lorentzian manifolds of constant sectional curvature will follow from the much more general results obtained in the next section.

6. Three-dimensional homogeneous contact Lorentzian manifolds

To complete the classification of contact pseudo-metric manifolds of constant sectional curvature and to find some relevant non-Sasakian examples, we shall classify all three-dimensional homogeneous contact pseudo-metric manifolds.

In the previous section, we concluded that in dimension $2n+1 \geqslant 5$ there are not flat contact Lorentzian structures. We now exhibit a three-dimensional flat contact metric structure.

Example 6.1 *(A flat contact Lorentzian structure on $\mathbb{R}^3$).* On $\mathbb{R}^3(x, y, z)$, we consider the Lorentzian metric defined by

$$g = \frac{1}{4}dx^2 - dy \otimes dz, \quad \text{that is,} \quad (g_{ij}) = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix} \tag{6.1}$$

and the 1-form

$$\eta = \frac{1}{2}\left(e^x dy + e^{-x} dz\right).$$

We easily get

$$d\eta = \frac{1}{4}\begin{pmatrix} 0 & e^x & -e^{-x} \\ -e^x & 0 & 0 \\ e^{-x} & 0 & 0 \end{pmatrix} \tag{6.2}$$

and $\eta \wedge d\eta \neq 0$. Thus, η is a contact form. The associated Reeb vector field ξ, completely determined by the conditions $\eta(\xi) = 1$ and $(d\eta)(\xi, \cdot) = 0$, is given by

$$\xi = e^{-x}\partial_y + e^x\partial_z.$$

Then, (6.1) implies that $g(\xi, \xi) = -1$, the contact distribution is spanned by vector fields

$$E_1 = e^{-x}\partial_y - e^x\partial_z, \qquad E_2 = 2\partial_x$$

and $\{\xi, E_1, E_2\}$ is a pseudo-orthonormal basis for g. Next, we consider the tensor φ of type $(1,1)$ defined, with respect to the basis $\{\partial_x, \partial_y, \partial_z\}$, by

$$\varphi = \begin{pmatrix} 0 & e^x & -e^{-x} \\ -e^{-x}/2 & 0 & 0 \\ e^x/2 & 0 & 0 \end{pmatrix}. \tag{6.3}$$

From (6.1), (6.2) and (6.3), we get $d\eta = g(\cdot, \varphi)$ and $\{\xi, E_1, E_2\}$ is a φ-basis with $E_2 = \varphi E_1$. Therefore, (η, φ, ξ, g) is a contact Lorentzian structure on $\mathbb{R}^3$. Since $\{\xi, E_1, E_2\}$ is a pseudo-orthonormal basis and the Lie brackets are given by

$$[\xi, E_1] = 0, \qquad [\xi, E_2] = 2E_1, \qquad [E_1, E_2] = -2\xi, \tag{6.4}$$

we find that the only non-vanishing covariant derivative of the Levi-Civita connection ∇ of (M, g) are

$$\nabla_{E_2}\xi = -2E_1, \qquad \nabla_{E_2}E_1 = 2\xi.$$

Thus, the curvature tensor satisfies $R(E_1, E_2) = R(E_2, \xi) = R(\xi, E_1) = 0$, that is, (η, g) is a flat contact Lorentzian structure on $\mathbb{R}^3$. Note that ξ is not a Killing vector field, as $\nabla_{E_2}\xi = -2E_1$. Moreover, from (6.4) we obtain $hE_1 = E_1$ and $hE_2 = -E_2$.

We now turn our attention to three-dimensional locally symmetric contact Lorentzian spaces. We recall that locally symmetric contact Riemannian three-spaces were classified in [5]. We have the following.

Theorem 6.2. *A three-dimensional locally symmetric contact Lorentzian manifold (M, η, g) is either flat or of constant sectional curvature $k = -1 = g(\xi, \xi)$.*

The following result is an immediate consequence of Theorem 6.2 and a result of [18].

Corollary 6.3. *The pseudo-Euclidean space $\mathbb{R}^3_1$ and the universal covering of the pseudo-hyperbolic space $\mathbb{H}^3_1(-1)$, are the only three-dimensional simply connected symmetric contact Lorentzian manifolds.*

To prove Theorem 6.2, one starts from the following classification result.

Theorem 6.4. *(See [7].) A three-dimensional Lorentzian locally symmetric space is locally isometric to either*

(i) *a Lorentzian space form $\mathbb{S}^3_1$, $\mathbb{R}^3_1$ or $\mathbb{H}^3_1$;*
(ii) *a direct product $\mathbb{R} \times \mathbb{S}^2_1$, $\mathbb{R} \times \mathbb{H}^2_1$, $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$; or*
(iii) *a symmetric space admitting a parallel null (that is, light-like) vector field.*

As proved in [8], cases (ii), (iii) of Theorem 6.4 only occur when the compatible Lorentzian metric is flat. So, if (M^3, η, g) is a locally symmetric contact Lorentzian space, then (M^3, g) has constant sectional curvature k. An argument very similar to the one developed for the Riemannian case in [5] then shows that $k = -1$ and this ends the proof of Theorem 6.2.

We now consider homogeneous contact Lorentzian three-manifolds. Similarly to the Riemannian case, a contact manifold (M, η) is said to be *homogeneous* if there exists a connected Lie group G of diffeomorphisms acting transitively on M and leaving η invariant. If a Lorentzian metric g satisfies (2.3) and G is a group of isometries, then (M, η, g) is said to be a *homogeneous contact Lorentzian manifold*.

Let $(M, \eta, g_L, \xi, \varphi)$ be a simply connected homogeneous contact Lorentzian three-manifold, with ξ time-like, and consider the Riemannian metric g determined by g_L and η via (4.1), that is, $g = g_L + 2\eta \otimes \eta$. Since both g_L and η are invariant under the action of the group G, so is the contact Riemannian structure (η, g, ξ, φ) on M. Therefore, three-dimensional homogeneous contact Lorentzian structures are in a one-to-one correspondence with the Riemannian ones via (4.1). For this reason, we start from the classification of three-dimensional homogeneous contact Riemannian structures, obtained in [14].

As proved in [14], if (M, η, g) is a simply connected three-dimensional homogeneous contact Riemannian manifold, then $M = G$ is a Lie group and the contact metric structure (η, g, ξ, φ) is left-invariant. In turn, this implies at once that $(\eta, g_L, \xi, \varphi)$ is also left-invariant. The classification obtained in [14] can be restated in the following form, using the scalar curvature r and the torsion invariant $\|\tau\| = 2\sqrt{\operatorname{tr} h^2}$ introduced by Chern and Hamilton in [9].

- *Sasakian case* $(\tau = 0)$.

(1) If G is unimodular, then it is (a) the Heisenberg group H_3 when $r = -2$; (b) the 3-sphere group $SU(2)$ when $r > -2$; (c) the group $\widetilde{SL}(2, R)$ when $r < -2$.

(2) If G is non-unimodular, its Lie algebra is given by $[e_1, e_2] = \alpha e_2 + 2\xi$, $[e_1, \xi] = [e_2, \xi] = 0$, where $\alpha \neq 0$. In this case, $r = -2\alpha^2 - 2 < -2$.

- *Non-Sasakian case* ($\|\tau\| \neq 0$ *is a constant*).

(1) If G is unimodular, then it is (a) $SU(2)$ when $r > -2(1 - \frac{\|\tau\|}{2\sqrt{2}})^2$; (b) the group $\tilde{E}(2)$ (universal covering of the group of rigid motions of Euclidean 2-space) when $r = -2(1 - \frac{\|\tau\|}{2\sqrt{2}})^2$; (c) $\widetilde{SL}(2, R)$ when $-2(1 + \frac{\|\tau\|}{2\sqrt{2}})^2 \neq r < -2(1 - \frac{\|\tau\|}{2\sqrt{2}})^2$; (d) the group $E(1, 1)$ (of rigid motions of Minkowski 2-space) when $r = -2(1 + \frac{\|\tau\|}{2\sqrt{2}})^2$.

(2) If G is non-unimodular, its Lie algebra is given by $[e_1, e_2] = \alpha e_2 + 2\xi$, $[e_1, \xi] = \gamma e_2$, $[e_2, \xi] = 0$, where $\alpha \neq 0$. In this case, $r < -2(1 - \frac{\|\tau\|}{2\sqrt{2}})^2$.

As we proved in Proposition 4.3, (η, g) is Sasakian if and only if so is (η, g_L). Moreover, (4.7) implies that the scalar curvatures r and r_L of g and g_L respectively are related by

$$r_L = r + 4 + 2 \operatorname{tr} h^2 = r + 4 + \frac{\|\tau\|^2}{2}. \tag{6.5}$$

By (6.5), in the Sasakian case we get that $r = -2$ if and only if $r_L = 2$, while in the non-Sasakian case we have

$$r > -2\left(1 - \frac{\|\tau\|}{2\sqrt{2}}\right)^2 \quad \Longleftrightarrow \quad r_L > +2\left(1 + \frac{\|\tau\|}{2\sqrt{2}}\right)^2$$

and

$$r \neq -2\left(1 + \frac{\|\tau\|}{2\sqrt{2}}\right)^2 \quad \Longleftrightarrow \quad r_L \neq +2\left(1 - \frac{\|\tau\|}{2\sqrt{2}}\right)^2.$$

Thus, the classification above leads at once to the following.

Theorem 6.5. *There is a one-to-one correspondence between homogeneous contact Riemannian three-manifolds and homogeneous contact Lorentzian three-manifolds. A simply connected homogeneous contact Lorentzian three-manifold is a Lie group G equipped with a left-invariant contact Lorentzian structure $(\varphi, \xi, \eta, g_L)$. More precisely, one of the following cases occurs*:

- *Sasakian case* $(\tau = 0)$.
  (1) *If G is unimodular, then it is*
    (i) *the Heisenberg group H_3 when $r_L = +2$;*
    (ii) *the 3-sphere group $SU(2)$ when $r_L > +2$;*
    (iii) *$\widetilde{SL}(2, R)$ when $r_L < 2$.*
  (2) *If G is non-unimodular, then its Lie algebra is given by*

$$[e_1, e_2] = \alpha e_2 + 2\xi, \qquad [e_1, \xi] = [e_2, \xi] = 0,$$

where $\alpha \neq 0$. In this case, $r_L = -2\alpha^2 + 2 < 2$.

- *Non-Sasakian case* ($\|\tau\| \neq 0$ *is a constant*).
  (1) *If G is unimodular, then it is*
    (i) $SU(2)$ *when* $r_L > 2(1+\frac{\|\tau\|}{2\sqrt{2}})^2$;
    (ii) $\tilde{E}(2)$ *when* $r_L = 2(1+\frac{\|\tau\|}{2\sqrt{2}})^2$;
    (iii) $\tilde{SL}(2,R)$ *when* $2(1-\frac{\|\tau\|}{2\sqrt{2}})^2 \neq r_L < 2(1+\frac{\|\tau\|}{2\sqrt{2}})^2$;
    (iv) $E(1,1)$ *when* $r_L = 2(1-\frac{\|\tau\|}{2\sqrt{2}})^2$.
  (2) *If G is non-unimodular, then its Lie algebra is given by*

$$[e_1,e_2] = \alpha e_2 + 2\xi, \qquad [e_1,\xi] = \gamma e_2, \qquad [e_2,\xi] = 0,$$

  where $\alpha \neq 0$. *In this case,* $r_L < 2(1+\frac{\|\tau\|}{2\sqrt{2}})^2$.

The classification given in Theorem 6.5 yields at once the following.

Corollary 6.6. *The three-sphere group* $SU(2)$ *is the only simply connected three-manifold which admits a homogeneous contact Lorentzian metric with scalar curvature* $r_L > 2(1+\frac{\|\tau\|}{2\sqrt{2}})^2$.

Next, Example 6.1 describes explicitly a flat left-invariant contact Lorentzian structure on the Lie group $E(1,1)$. Three-dimensional Lie groups admitting a flat left-invariant Lorentzian metric were classified by the present author in [6] (see also [12,15]). Comparing such classification with Theorem 6.5, we get the following.

Corollary 6.7. *The Lie group* $E(1,1)$ *is the only simply connected three-manifold which admits a flat homogeneous contact Lorentzian metric.*

Note that, correspondingly, Theorem 3.1 of [14] yields that the universal covering of the Lie group $E(2)$ is the only simply connected three-manifold which admits a flat homogeneous contact Riemannian metric.

References

[1] A. Bejancu, K.L. Duggal, Real hypersurfaces of indefinite Kaehler manifolds, Int. J. Math. Math. Sci. 16 (1993) 545–556.
[2] D.E. Blair, Two remarks on contact metric structures, Tôhoku Math. J. 29 (1977) 319–324.
[3] D.E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Progr. Math., Birkhäuser, Boston, 2002.
[4] D.E. Blair, D. Perrone, Conformally Anosov flows in contact metric geometry, Balkan J. Geom. Appl. 3 (1998) 33–46.
[5] D.E. Blair, R. Sharma, Three-dimensional locally symmetric contact metric manifolds, Boll. Unione Mat. Ital. (7) 4-A (1990) 385–390.
[6] G. Calvaruso, Homogeneous structures on three-dimensional Lorentzian manifolds, J. Geom. Phys. 57 (2007) 1279–1291; G. Calvaruso, J. Geom. Phys. 58 (2008) 291–292, Addendum.
[7] G. Calvaruso, Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds, Geom. Dedicata 127 (2007) 99–119.
[8] G. Calvaruso, D. Perrone, Contact pseudo-metric manifolds, Differential Geom. Appl. 28 (2010) 615–634.
[9] S.S. Chern, R.S. Hamilton, On Riemannian metrics adapted to three-dimensional contact manifolds, in: Lecture Notes in Math., vol. 1111, Springer-Verlag, Berlin, 1985, pp. 279–305.
[10] K.L. Duggal, Space time manifolds and contact structures, Int. J. Math. Math. Sci. 13 (1990) 545–554.
[11] B. O'Neill, Semi-Riemannian Geometry, Academic Press, New York, 1983.
[12] K. Nomizu, Left-invariant Lorentz metrics on Lie groups, Osaka J. Math. 16 (1979) 143–150.
[13] Z. Olszak, On contact metric manifolds, Tôhoku Math. J. 31 (1979) 247–253.
[14] D. Perrone, Homogeneous contact Riemannian three-manifolds, Illinois J. Math. 42 (1998) 243–256.
[15] S. Rahmani, Métriques de Lorentz sur les groupes de Lie unimodulaires de dimension trois, J. Geom. Phys. 9 (1992) 295–302.
[16] S. Sasaki, Y. Hatakeyama, On differentiable manifolds with certain structures which are closely related to almost contact structures II, Tôhoku Math. J. 13 (1961) 281–294.
[17] S. Sasaki, Y. Hatakeyama, On differentiable manifolds with contact metric structures, J. Math. Soc. Japan 14 (1962) 249–271.
[18] T. Takahashi, Sasakian manifold with pseudo-Riemannian metrics, Tôhoku Math. J. 21 (1969) 271–290.
[19] S. Tanno, The topology of contact Riemannian manifolds, Illinois J. Math. 12 (1968) 700–717.